On the local structure of Lorentzian Einstein manifolds with parallel distribution of null lines

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Abstract. We study transformations of coordinates on a Lorentzian Einstein manifold with a parallel distribution of null lines and show that the general Walker coordinates can be simplified. In these coordinates, the full Lorentzian Einstein equation is reduced to equations on a family of Einstein Riemannian metrics.

Dedicated to Dmitri Vladimirovich Alekseevsky on his 70th birthday

1. Introduction and statement of results

Recently in [**15**] G.W. Gibbons and C.N. Pope considered the Einstein equation on Lorentzian manifolds with holonomy algebras contained in $\mathfrak{sim}(n)$. A Lorentzian manifold (M,g) of dimension $n+2$ has holonomy algebra contained in $\mathfrak{sim}(n)$ if and only if it admits a parallel distribution l of null lines (l is a vector subbundle of rank one of the tangent bundle of M such that it holds $g(X,X)=0$ and $\nabla_Y X$ is a section of l for all sections X of l and vector fields Y on M, here ∇ is the Levi-Civita connection defined by g). Lorentzian manifolds with this property have special Lorentzian holonomy and are of interest both in geometry (e.g. [**1, 2, 4, 6, 25, 28**]) and theoretical physics (e.g. [**5, 7, 8, 9, 16**]). Any such manifold admits local coordinates $x^+, x^1, ..., x^n, x^-$, the so-called *Walker coordinates*, such that the metric g has the form

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + h + 2A\mathrm{d}x^- + H(\mathrm{d}x^-)^2, \tag{1}$$

where $h = h_{ij}(x^1, ..., x^n, x^-)\mathrm{d}x^i\mathrm{d}x^j$ is an x^--dependent family of Riemannian metrics, $A = A_i(x^1, \dots, x^n, x^-)\ \mathrm{d}x^i$ is an x^--dependent family of one-forms, and H is a local function on M, [**28**]. The vector field $\partial_+ := \frac{\partial}{\partial x^+}$ defines the parallel distribution of null lines. We assume that the indices $i, j, k, \dots$ run from 1 to n, and the indices $a, b, c, \dots$ run in $+, 1, \dots, n, -$ and we use the Einstein convention for sums. Furthermore, given coordinates $(x^+, x^1, \dots, x^n, x^-)$ or $(\tilde{x}^+, \tilde{x}^1, \dots, \tilde{x}^n, \tilde{x}^-)$ we write $\partial_a := \frac{\partial}{\partial x^a}$ and $\tilde{\partial}_a := \frac{\partial}{\partial \tilde{x}^a}$.

The Einstein equation is the fundamental equation of General Relativity. In the absence of matter it has the form

$$\mathrm{Ric} = \Lambda g, \tag{2}$$

where g is a Lorentzian metric on a manifold M, Ric is the Ricci tensor of the metric g, i.e. $\mathrm{Ric}_{ab} = R^c{}_{abc}$, where R is the curvature tensor of the metric g, and $\Lambda \in \mathbb{R}$ is the cosmological, or Einstein constant. If a metric g of a smooth manifold (M,g) satisfies this equation, then (M,g) is called *an Einstein manifold*. If moreover $\Lambda = 0$, then it is called *vacuum Einstein* or *Ricci-flat*. In dimension 4 examples of Einstein metrics are constructed in [**17, 18, 19, 23, 24, 27**].

We assume that $n \geq 2$, since for $n = 0$ the problem is trivial and for $n = 1$ the metric (1) cannot be non-flat and Einstein [**15, 13**].

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In [**15**] it is shown that the Einstein equation for a Lorentzian metric of the form (1) implies

$$H = \Lambda \cdot (x^+)^2 + x^+ H_1 + H_0, \tag{3}$$

where H_0 and H_1 do not depend on x^+. Furthermore, in [**15**] it is proved that Equation (2) is equivalent to Equation (3) and the following system of equations

$$\begin{aligned}\Delta H_0 - \frac{1}{2}F^{ij}F_{ij} - 2A^i\partial_i H_1 - H_1\nabla^i A_i + 2\Lambda A^i A_i - 2\nabla^i \dot{A}_i &\\ +\frac{1}{2}\dot{h}^{ij}\dot{h}_{ij} + h^{ij}\ddot{h}_{ij} + \frac{1}{2}h^{ij}\dot{h}_{ij}H_1 &= 0, \qquad (4)\\ \nabla^j F_{ij} + \partial_i H_1 - 2\Lambda A_i + \nabla^j \dot{h}_{ij} - \partial_i(h^{jk}\dot{h}_{jk}) &= 0, \qquad (5)\\ \Delta H_1 - 2\Lambda\nabla^i A_i + \Lambda h^{ij}\dot{h}_{ij} &= 0, \qquad (6)\\ \mathrm{Ric}_{ij} &= \Lambda h_{ij}, \qquad (7)\end{aligned}$$

where $\Delta H_0 = h^{ij}(\partial_i\partial_j H_0 - \Gamma^k_{ij}\partial_k H_0)$ is the Laplace-Beltrami operator of the metrics $h(x^-)$ applied to H_0, $F_{ij} = \partial_i A_j - \partial_j A_i$ are the components of the differential of the one-form $A(x^-) = A_i \mathrm{d}x^i$. A dot denotes the derivative with respect to x^- and $\nabla^i \dot{A}_i = (\nabla_{\partial_i}(\dot{A})^\#)^i$ is the divergence w.r.t. $h(x^-)$ of $\dot{A}$.

Of course, the Walker coordinates are not defined canonically and any other Walker coordinates $\tilde{x}^+, \tilde{x}^1, \ldots, \tilde{x}^n, \tilde{x}^-$ such that $\tilde{\partial}_+ = \partial_+$ are given by the following transformation (see [**25**] and Section 3)

$$\tilde{x}^+ = x^+ + \varphi(x^1, ..., x^n, x^-), \quad \tilde{x}^i = \psi^i(x^1, ..., x^n, x^-), \quad \tilde{x}^- = x^- + c.$$

Now, the aim of the paper is to simplify these coordinates on Einstein manifolds and, in consequence, find easier equivalences to the Einstein equation when written in the new coordinates. That the coordinates can be simplified in special situations was already shown in [**25**]:

Proposition 1 (Schimming [**25**]). *Let (M, g) be a Lorentzian manifold with a parallel null vector field. Then there exist local coordinates $\big(U, (x^+, x^1, \ldots, x^n, x^-)\big)$ such that the metric is given as*

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + h_{kl}\mathrm{d}x^k\mathrm{d}x^l$$

with h_{kl} smooth functions on U with $\partial_+ h_{kl} = 0$.

Note that the condition for (M, g) to admit a parallel null vector field is stronger then the condition to admit a parallel distribution of null lines. The first result of the present paper generalises Proposition 1 to manifolds with only a parallel distribution of null lines:

Theorem 1. *Let (M, g) be a Lorentzian manifold with a parallel distribution of null lines. Then there exist local coordinates $\big(U, (x^+, x^1, \ldots, x^n, x^-)\big)$ such that the metric is given as*

$$g = \left(2\mathrm{d}x^+ + H\mathrm{d}x^-\right)\mathrm{d}x^- + h_{kl}\mathrm{d}x^k\mathrm{d}x^l$$

with H and h_{kl} smooth functions on U with $\partial_+ h_{kl} = 0$.

With respect to coordinates as in Theorem 1 the Einstein equations (4–7) become much easier:

$$\begin{aligned}\Delta H_0 + \frac{1}{2}\dot{h}^{ij}\dot{h}_{ij} + h^{ij}\ddot{h}_{ij} + \frac{1}{2}h^{ij}\dot{h}_{ij}H_1 &= 0, \qquad (8)\\ \partial_i H_1 + \nabla^j \dot{h}_{ij} - \partial_i(h^{jk}\dot{h}_{jk}) &= 0, \qquad (9)\\ \Delta H_1 + \Lambda h^{ij}\dot{h}_{ij} &= 0, \qquad (10)\\ \mathrm{Ric}_{ij} &= \Lambda h_{ij}. \qquad (11)\end{aligned}$$

Then we assume that the manifold is Einstein, and, based on Equation (3), we prove the following:

Theorem 2. *Let (M,g) be a Lorentzian manifold with a parallel distribution of null lines and assume that M is Einstein with Einstein constant Λ. Then there exist local coordinates $\left(x^+,x^1,\dots,x^n,x^-\right)$ such that the metric is given as*

$$g=\left(2\mathrm{d}x^+ + (\Lambda(x^+)^2+x^+H_1)\mathrm{d}x^-\right)\mathrm{d}x^- + h_{kl}\mathrm{d}x^k\mathrm{d}x^l$$

with H_1 and h_{kl} smooth functions on U with $\partial_+h_{kl}=\partial_+H_1=0$ and satisfying the equations

$$\frac{1}{2}\dot h^{ij}\dot h_{ij}+h^{ij}\ddot h_{ij}+\frac{1}{2}h^{ij}\dot h_{ij}H_1=0, \tag{12}$$

$$\partial_iH_1+\nabla^j\dot h_{ij}-\partial_i(h^{jk}\dot h_{jk})=0, \tag{13}$$

$$\Delta H_1+\Lambda h^{ij}\dot h_{ij}=0, \tag{14}$$

$$\mathrm{Ric}_{ij}=\Lambda h_{ij}. \tag{15}$$

Conversely, any such metric is Einstein with Einstein constant Λ.

Note that if (M,g) admits a parallel null vector field, then the Walker coordinates in (1) satisfy $\partial_+H=0$ and we get Proposition 1 from Theorem 1 (see Remark 1 below). If, in addition, such a metric is Einstein, then $\Lambda=0$, i.e. this metric is Ricci-flat and the equations (4–7) take the following more simplified form

$$\frac{1}{2}\dot h^{ij}\dot h_{ij}+h^{ij}\ddot h_{ij}=0, \tag{16}$$

$$\nabla^j\dot h_{ij}-\partial_i(h^{jk}\dot h_{jk})=0, \tag{17}$$

$$\mathrm{Ric}_{ij}=0. \tag{18}$$

Finally, as the main result of the paper we show that in the the case $\Lambda\neq 0$ we can do better.

Theorem 3. *Let (M,g) be a Lorentzian manifold of dimension $n+2$ admitting a parallel distribution of null lines. If (M,g) is Einstein with the non-zero cosmological constant Λ then there exist local coordinates $\left(x^+,x^1,\dots,x^n,x^-\right)$ such that the metric g has the form*

$$g=2\mathrm{d}x^+\mathrm{d}x^-+h_{kl}\mathrm{d}x^k\mathrm{d}x^l+(\Lambda(x^+)^2+H_0)(\mathrm{d}x^-)^2$$

with $\partial_+h_{kl}=\partial_+H_0=0$, h_{kl} defines an x^--dependent family of Riemannian Einstein metrics with the cosmological constant Λ, satisfying the equations

$$\Delta H_0+\frac{1}{2}h^{ij}\ddot h_{ij}=0, \tag{19}$$

$$\nabla^j\dot h_{ij}=0, \tag{20}$$

$$h^{ij}\dot h_{ij}=0, \tag{21}$$

$$\mathrm{Ric}_{ij}=\Lambda h_{ij}, \tag{22}$$

where $\dot h_{ij}=\partial_-h_{ij}$. Conversely, any such metric is Einstein.

Remark that in [**15**] it is shown that Equation (6) follows from (5) and (7), i.e. it may be omitted from the Einstein equation. By the same reason Equations (10) (14) and (21) may be omitted. On the other hand, these equations can be used as the corollaries of the Einstein equation.

Thus, we reduce the Einstein equation with $\Lambda\neq 0$ on a Lorentzian manifold with holonomy algebra contained in $\mathfrak{sim}(n)$ to the study of families of Einstein Riemannian metrics satisfying Equation (20). If $\Lambda=0$ and $\partial_+H\neq 0$, i.e. $H_1\neq 0$, then consider the coordinates as in Theorem 2. Equation (14) shows that H_1 is a family of harmonic functions on the family of the Riemannian manifolds with metrics $h(x^-)$. Fixing any such H_1 we get Equations (12) and (13) on the family of Ricci-flat Riemannian metrics $h(x^-)$. Finally, if (M,g) is Einstein and it admits a parallel null vector field, then it is Ricci flat and this is equivalent to Equations (16) and (17) on the family of Ricci-flat Riemannian metrics $h(x^-)$. In Section 2 we consider the holonomy algebra of (M,g) and the de Rham decomposition for the family of Riemannian metrics $h(x^-)$.

Note that to find the required transformation of the coordinates in Theorem 3 we need to solve a system of ODE's, while in [**25**] several PDE's need to be solved.

Examples of Einstein metrics of the form (1) with h independent of x^- and each possible holonomy algebra are constructed in [**13**]. It is interesting to construct examples of Einstein manifolds satisfying some global properties, e.g. global hyperbolic, as in [**1**] or [**2**].

In Section 4 we consider examples in dimension 4.

2. Consequences

Let us consider some consequences of the above theorems. Let (M,g) be a Lorentzian manifold with a parallel distribution of null lines. Without loss of generality we may assume that (M,g) is locally indecomposable, i.e. locally it is not a product of a Lorentzian and of a Riemannian manifold. The holonomy of such manifolds are contained in $\mathfrak{sim}(n)=(\mathbb{R}\oplus\mathfrak{so}(n))\ltimes\mathbb{R}^n$. In [**22**] it was shown that the projection $\mathfrak{h}$ of the holonomy algebra of (M,g) onto $\mathfrak{so}(n)$ has to be a Riemannain holonomy algebra. Now, recall that for each Riemannian holonomy algebra $\mathfrak{h}\subset\mathfrak{so}(n)$ there exists a decomposition

$$\mathbb{R}^n=\mathbb{R}^{n_0}\oplus\mathbb{R}^{n_1}\oplus\cdots\oplus\mathbb{R}^{n_r} \tag{23}$$

and the corresponding decomposition into the direct sum of ideals

$$\mathfrak{h}=\{0\}\oplus\mathfrak{h}_1\oplus\cdots\oplus\mathfrak{h}_r \tag{24}$$

such that each $\mathfrak{h}_\alpha\subset\mathfrak{so}(n_\alpha)$ is an irreducible Riemannian holonomy algebra, in particular it coincides with one of the following subalgebras of $\mathfrak{so}(n_\alpha)$: $\mathfrak{so}(n_\alpha)$, $\mathfrak{u}(\frac{n_\alpha}{2})$, $\mathfrak{su}(\frac{n_\alpha}{2})$, $\mathfrak{sp}(\frac{n_\alpha}{4})\oplus\mathfrak{sp}(1)$, $\mathfrak{sp}(\frac{n_\alpha}{4})$, $G_2\subset\mathfrak{so}(7)$, $\mathfrak{spin}_7\subset\mathfrak{so}(8)$ or it is an irreducible symmetric Berger algebra (i.e. it is the holonomy algebra of a symmetric Riemannian manifold and it is different from $\mathfrak{so}(n_\alpha)$, $\mathfrak{u}(\frac{n_\alpha}{2})$, $\mathfrak{sp}(\frac{n_\alpha}{4})\oplus\mathfrak{sp}(1)$). Recall that if the holonomy algebra of a Riemannian manifold is a symmetric Berger algebra, then the manifold is locally symmetric.

In [**11, 13**] it is proven that if (M,g) is Einstein with $\Lambda\neq 0$, then the holonomy algebra of (M,g) has the form $\mathfrak{g}=(\mathbb{R}\oplus\mathfrak{h})\ltimes\mathbb{R}^n$, moreover, each subalgebra $\mathfrak{h}_\alpha\subset\mathfrak{so}(n_\alpha)$ from the decomposition (24) coincides with one of the algebras $\mathfrak{so}(n_\alpha)$, $\mathfrak{u}(\frac{n_\alpha}{2})$, $\mathfrak{sp}(\frac{n_\alpha}{4})\oplus\mathfrak{sp}(1)$ or with a symmetric Berger algebra, and in the decomposition (23) it holds $n_0=0$. Next, if $\Lambda=0$, then one of the following holds:

(A) $\mathfrak{g}=(\mathbb{R}\oplus\mathfrak{h})\ltimes\mathbb{R}^n$ and at least one of the subalgebras $\mathfrak{h}_\alpha\subset\mathfrak{so}(n_\alpha)$ from the decomposition (24) coincides with one of the algebras $\mathfrak{so}(n_\alpha)$, $\mathfrak{u}(\frac{n_\alpha}{2})$, $\mathfrak{sp}(\frac{n_\alpha}{4})\oplus\mathfrak{sp}(1)$ or with a symmetric Berger algebra.

(B) $\mathfrak{g}=\mathfrak{h}\ltimes\mathbb{R}^n$ and each subalgebra $\mathfrak{h}_\alpha\subset\mathfrak{so}(n_\alpha)$ from the decomposition (24) coincides with one of the algebras $\mathfrak{su}(\frac{n_\alpha}{2})$, $\mathfrak{sp}(\frac{n_\alpha}{4})$, $G_2\subset\mathfrak{so}(7)$, $\mathfrak{spin}_7\subset\mathfrak{so}(8)$.

In [**4**] it is proved that there exist Walker coordinates $x^+,x_0^1,\dots,x_0^{n_0},\dots,x_r^1,...,x_r^{n_r},x^-$ that are adapted to the decomposition (24). This means that $h=h_0+h_1+\cdots+h_r$, $h_0=\sum_{i=1}^{n_0}(\mathrm{d}x_0^i)^2$ and $A=\sum_{\alpha=1}^r\sum_{k=1}^{n_\alpha}A_k^\alpha\mathrm{d}x_\alpha^k$ and for each $1\leq\alpha\leq r$ it holds $h_\alpha=\sum_{i,j=1}^{n_\alpha}h_{\alpha ij}\mathrm{d}x_\alpha^i\mathrm{d}x_\alpha^j$ with $\frac{\partial}{\partial x_\beta^k}h_{\alpha ij}=\frac{\partial}{\partial x_\beta^k}A_i^\alpha=0$ for all $1\leq i,j\leq n_\alpha$ if $\beta\neq\alpha$. We will show that the transformations can be chosen in such a way that the new coordinates are adapted in this sense.

Proposition 2. *Let (M,g) be a Lorentzian manifold with a parallel distribution of null lines and let $\mathfrak{h}$ be the projection of its holonomy algebra onto $\mathfrak{so}(n)$ decomposed as in* (24)*.*

(1) Then the coordinates found in Theorem 1 can be chosen to be adapted to this decomposition.

(2) If (M,g) is Einstein with $\Lambda\neq 0$, then there exist coordinates adapted to this decomposition with the properties as in Theorem 3 and with $n_0=0$.

We will prove this proposition in the next section. It shows that the Einstein conditions written as in the formulae after Theorem 1 and in Theorem 3 can, in addition, be formulated in adapted coordinates.

Now we discuss to which extend the Einstein equations in the theorems have to be satisfied for each of the h_α's separately when written in the coordinates of Proposition 2. First, let $\Lambda\neq 0$ and consider (19 – 22). It is obvious that each h_α satisfies (20) and (22). Using the first variation formula for the Ricci tensor (see e.g. [**3**, Theorem 1.174]), in [**15**] it is shown that (6) follows from

(5) and (7) by taking the divergence of (5). Hence, using the divergence with respect to the metric h_α, (20) and (22) imply that each h_α satisfies also (21). This means that one has to solve (20 – 22) separately for each h_α and then find H_0 from (19).

Similarly, if $\Lambda = 0$, consider (8 – 11). Obviously, each h_α has to be Ricci-flat. Applying the divergence with respect to h_α to (13) we get that $\Delta_\alpha H_1 = 0$. This together with (13) shows that $H_1 = \sum_\alpha H_{1\alpha}$, where each $H_{1\alpha}$ depends only on x^i_α and it is harmonic with respect to h_α. Now each h_α satisfies (9) with H_1 replaced by $H_{1\alpha}$.

Next we study the possible summands in the decomposition (24) under the assumption that the manifold (M, g) is Einstein with $\Lambda \neq 0$. First we claim that if $\mathfrak{h}_\alpha \subset \mathfrak{so}(n_\alpha)$ is a symmetric Berger algebra, then each metric in the family $h_\alpha(x^-)$ is locally symmetric and its holonomy algebra coincides with $\mathfrak{h}_\alpha$. Indeed, the holonomy algebra $\mathfrak{h}_\alpha(x^-)$ of each metric in the family $h_\alpha(x^-)$ is contained in $\mathfrak{h}_\alpha$ and it is non-trivial due to (22). Since $\mathfrak{h}_\alpha \subset \mathfrak{so}(n_\alpha)$ is a symmetric Berger algebra its space of curvature tensors $\mathcal{R}(\mathfrak{h}_\alpha)$ is one-dimensional. This shows that $\mathfrak{h}_\alpha(x^-) = \mathfrak{h}_\alpha$. If $\mathfrak{h}_\alpha = \mathfrak{u}(\frac{n_\alpha}{2})$ (resp., $\mathfrak{h}_\alpha = \mathfrak{sp}(\frac{n_\alpha}{4}) \oplus \mathfrak{sp}(1)$), then each metric in the family $h_\alpha(x^-)$ is Kähler-Einstein (resp., quaternionic-Kähler). For some values of x^- the metric $h_\alpha(x^-)$ can be decomposable, but it does not contain a flat factor. If $\mathfrak{h}_\alpha = \mathfrak{so}(n_\alpha)$, then we get a general family of Einstein metrics. For some values of x^- the metric $h_\alpha(x^-)$ can be decomposable, but it does not contain a flat factor.

Proposition 3. *Under the current assumptions, if $\mathfrak{h}_\alpha \subset \mathfrak{so}(n_\alpha)$ is a symmetric Berger algebra, then h_α satisfies the equation*

$$\nabla_i(h^{kt}_\alpha \dot{h}_{\alpha tj}) - 2\dot{\Gamma}^k_{ij} = 0, \quad 1 \leq i, j, k \leq n_\alpha, \tag{25}$$

where Γ^k_{ij} is the family of the Christoffel symbols for the family of the Riemannian metrics $h(x^-)$.

Note that the Equation (25) is stronger then the Equation (20), since the last equation is obtained from the first one by taking the trace. This proposition will be proved below.

Finally, suppose that $\Lambda = 0$ and the holonomy algebra $\mathfrak{g}$ of (M, g) is as in the case (A) above. Suppose that $\mathfrak{h}_\alpha$ is one of $\mathfrak{so}(n_\alpha)$, $\mathfrak{u}(\frac{n_\alpha}{2})$, $\mathfrak{sp}(\frac{n_\alpha}{4}) \oplus \mathfrak{sp}(1)$ or it is a symmetric Berger algebra. Equation (15) shows that in the first three cases each metric in the family h_α is Ricci-flat, consequently, its holonomy algebra is contained, respectively, in $\mathfrak{so}(n_\alpha)$, $\mathfrak{su}(\frac{n_\alpha}{2})$, $\mathfrak{sp}(\frac{n_\alpha}{4})$. If $\mathfrak{h}_\alpha$ is a symmetric Berger algebra, then by the same reasons each metric in the family $h_\alpha(x^-)$ is flat. Otherwise $\mathfrak{h}_\alpha$ is either trivial or it is one of $\mathfrak{su}(\frac{n_\alpha}{2})$, $\mathfrak{sp}(\frac{n_\alpha}{4})$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}_7 \subset \mathfrak{so}(8)$. Each metric in the family h_α is Ricci-flat and it has holonomy algebra contained in $\mathfrak{h}_\alpha$.

To sum up the consequences we remark that the problem of finding Einstein Lorentzian metrics with $\Lambda \neq 0$ is reduced first to the problem of finding families of Einstein Riemannian metrics satisfying Equation (20) (or (25) for the symmetric case) and then to Poisson equation (19) on the function H_0. This is related to the module spaces of Einstein metrics [**3**]. For example, for most of symmetric Berger algebras $\mathfrak{h}_\alpha \subset \mathfrak{so}(n_\alpha)$ it holds that h_α is an isolated metric, i.e. it is independent of x^- [**20, 21**]. Hence, since it is symmetric, it is uniquely defined by Λ. Similarly, if $\Lambda = 0$ and $\partial_+ H \neq 0$, i.e. $H_1 \neq 0$, then consider the coordinates as in Theorem 1. Equation (10) shows that H_1 is a family of harmonic functions on the family of the Riemannian manifolds with metrics $h(x^-)$. Fixing any such H_1 we get Equation (9) on the family of Ricci-flat Riemannian metrics $h(x^-)$ and then Poisson equation (8) on the function H_0. Finally, if (M, g) is Einstein and it admits a parallel null vector field, then it is Ricci-flat and this is equivalent to the equations (16) and (17) on the family of Ricci-flat Riemannian metrics $h(x^-)$.

3. Proofs

Coordinate transformations. In order to simplify the Walker coordinates, first we have to describe the most general coordinate transformation leaving the form (1) invariant. This was already done in [**25**] in the case of a parallel null vector field.

Proposition 4. *The most general coordinate transformation with $\tilde{\partial}_+ = \partial_+$ that preserves the form (1) is given by*

$$\tilde{x}^+ = x^+ + \varphi(x^1, ..., x^n, x^-), \quad \tilde{x}^i = \psi^i(x^1, ..., x^n, x^-), \quad \tilde{x}^- = x^- + c. \tag{26}$$

If the metric and its inverse is written as

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h & A \\ 1 & A^t & H \end{pmatrix} \quad \textit{and} \quad g^{-1} = \begin{pmatrix} F & B^t & 1 \\ B & h^{-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{27}$$

with $B = -h^{-1}A$ *and* $F + H + A^tB = 0$*, then in the new coordinates it holds*

$$\tilde{h}^{ij} = \partial_k \psi^i h^{kl} \partial_l \psi^j \tag{28}$$

$$\tilde{B}^i = \partial_- \psi^i + B^k \partial_k \psi^i + h^{kl} \partial_k \varphi \partial_l \psi^i \tag{29}$$

$$\tilde{F} = F + \partial_- \varphi + B^k \partial_k \varphi + h^{kl} \partial_k \varphi \partial_l \varphi. \tag{30}$$

Proof. Since $\tilde{\partial}_+ = \partial_+$, the transformation formula for the canonical basis implies

$$\partial_+ \tilde{x}^- = 0, \ \partial_+ \tilde{x}^k = 0, \ \text{and} \ \ \partial_+ \tilde{x}^+ = 1.$$

Furthermore we get

$$0 = g(\partial_+, \partial_i) = \partial_+ \tilde{x}^+ \partial_i \tilde{x}^- g(\tilde{\partial}_+, \tilde{\partial}_-) + \partial_+ \tilde{x}^+ \partial_i \tilde{x}^k g(\tilde{\partial}_+, \tilde{\partial}_k).$$

As we require $g(\tilde{\partial}_+, \tilde{\partial}_k) = 0$ this implies $\partial_i \tilde{x}^- = 0$. Finally we have to check

$$1 = g(\partial_+, \partial_-) = \partial_+ \tilde{x}^+ \partial_- \tilde{x}^-,$$

which implies $\partial_- \tilde{x}^- = 1$. This shows that the most general transformation is of the form (26).

In order to write down the inverse metric coefficients in the new coordinates first we see that in the coordinates (1) the metric and its inverse are given as in (27). The transformation formula for the inverse metric coefficients g^{ab} is given by

$$\partial_c \tilde{x}^a g^{cd} \partial_d \tilde{x}^b = \tilde{g}^{ab},$$

where a and b run over $+, 1, \ldots, n, -$. This implies that

$$\tilde{B}^i = \tilde{g}^{+i} = \partial_- \tilde{x}^i + B^k \partial_k \tilde{x}^i + h^{kl} \partial_k \tilde{x}^+ \partial_l \tilde{x}^i,$$

which is Equation (29). Furthermore we get

$$\tilde{F} = \tilde{g}^{++} = F + \partial_+ \tilde{x}^+ \partial_- \tilde{x}^+ + B^k \partial_+ \tilde{x}^+ \partial_k \tilde{x}^+ + h^{kl} \partial_k \tilde{x}^+ \partial_l \tilde{x}^+,$$

which is Equation (30). In the same way the equations for $\tilde{h}^{ij}$. □

Proof of Theorem 1. Setting $\tilde{B}^i$ to zero for each $i = 1, \ldots, n$ in the transformation formula above we obtain a linear PDE for the function ψ

$$\partial_- \psi = -\left(B^k + h^{kl} \partial_l \varphi \right) \partial_k \psi, \tag{31}$$

and we have to find n linear independent solutions $\psi^1, \ldots, \psi^n$. This problem can be solved for the following reasons: Fix the function $\varphi = \varphi(x^1, \ldots, x^n, x^-)$, e.g. $\varphi \equiv 0$, and consider the characteristic vector field of (31)

$$X := \partial_- + \left(B^k + h^{kl} \partial_l \varphi \right) \partial_k.$$

Obviously, Equation (31) is equivalent to the equation

$$X(\psi) = d\psi(X) = 0. \tag{32}$$

We have $[\partial_+, X] = 0$. Hence, we find coordinates $(y^+, y^1, \ldots, y^n, y^-)$ such that

$$\tfrac{\partial}{\partial y^+} = \partial_+ \ \ \text{and} \ \ \tfrac{\partial}{\partial y^-} = X.$$

Now, any function $\psi = \psi(y^1, \ldots, y^n)$ satisfies Equation (32). Note that $\partial_+ y^- = \partial_+ y^i = 0$ and therefore also $\partial_+ \psi = 0$. Taking n linear independent solutions gives us the required solutions ψ^i of Equation (31) to build the new coordinate system. □

Remark 1. In order to obtain Schimming's result of Proposition 1 one has to set $\tilde{H}$ to zero obtaining the additional equation

$$\partial_- \varphi = -F - B^k \partial_k \varphi - h^{kl} \partial_k \varphi \partial_l \varphi \tag{33}$$

together with the linear Equation (31). Although Equation (33) cannot be written in the form $X(\varphi) = 0$, it can be solved using characteristics (see below).

Remark 2. Note that Schimming's result cannot be true only with the assumption of a parallel distribution of null lines: Since in this case H and thus F may depend on x^+ but φ does not, Equation (33) cannot be solved. In other words, the x^+-dependence of H in general cannot be changed by these coordinate transformations. But in case of Einstein metrics with arbitrary Einstein constant Λ, Theorem 2 shows that one can get rid of the part of H that does not depend on x^+.

Proof of Theorem 2. We fix coordinates $(x^+, x^1, \dots, x^n, x^-)$ as in Theorem 1 with $A_i = 0$. Since (M, g) is Einstein it holds that

$$H = \Lambda (x^+)^2 + x^+ H_1 + H_0,$$

where $\partial_+ H_1 = \partial_+ H_0 = 0$. Now we try to find an appropriate coordinate transformation consisting of functions φ and ψ^i as in Proposition 4. First we consider the equation

$$\partial_- \varphi \;=\; H_0 - H_1 \varphi + \Lambda \varphi^2 - h^{kl} \partial_k \varphi \partial_l \varphi. \tag{34}$$

This equation can be solved by the method of characteristics (for details see for example [**26**, Chapter 10, Section 1]). Since the x^- derivative of φ is isolated, a characteristic is given by $(x^1, \dots, x^n) \mapsto (x^1, \dots x^n, 0)$ and the parameter of the characteristic curves can be chosen to be x^-. Let φ be a smooth solution of this equation. With respect to this φ we consider the equation

$$\partial_- \psi \;=\; -h^{kl} \partial_k \varphi \partial_l \psi. \tag{35}$$

As in Theorem 1, we find n linear independent solutions $\psi^1, \dots, \psi^n$ to this equation. Hence, in the new coordinates given as in (26) we still have $\tilde{B}^k = 0$. Now, since (M, g) is Einstein, it is

$$\tilde{H} \;=\; \Lambda (\tilde{x}^+)^2 + \tilde{x}^+ \tilde{H}_1 + \tilde{H}_0 \;=\; \Lambda (x^+)^2 + (2\Lambda \varphi + \tilde{H}_1) x^+ + \tilde{H}_1 \varphi + \Lambda \varphi^2 + \tilde{H}_0.$$

On the other hand, from the transformation formula and $\tilde{B}^k = 0$ we have

$$\begin{aligned} \tilde{H} \;=\; -\tilde{F} \;&=\; -F - \partial_- \varphi - h^{kl} \partial_k \varphi \partial_l \varphi \\ &=\; \big(\Lambda (x^+)^2 + x^+ H_1 + H_0 \big) - \partial_- \varphi - h^{kl} \partial_k \varphi \partial_l \varphi. \end{aligned}$$

Comparing these two equations and differentiating w.r.t. ∂_+ shows that $(2\Lambda \varphi + \tilde{H}_1) \;=\; H_1$ and furthermore

$$\Lambda \varphi^2 + \tilde{H}_0 + \tilde{H}_1 \varphi \;=\; H_0 - \partial_- \varphi - h^{kl} \partial_k \varphi \partial_l \varphi.$$

Hence, putting this together we get

$$\tilde{H}_0 \;=\; H_0 - \partial_- \varphi - h^{kl} \partial_k \varphi \partial_l \varphi + \Lambda \varphi^2 - H_1 \varphi.$$

But since φ satisfies Equation (34), we obtain $\tilde{H}_0 = 0$ in the new coordinates. □

Curvature tensors. For the proof of Theorem 3 we need some algebraic preliminaries. The tangent space to M at any point $m \in M$ can be identified with the Minkowski space $\mathbb{R}^{1,n+1}$. Denote by g the metric on it. Let $\mathbb{R}p$ be the null line corresponding to the parallel distribution. Let $\mathcal{R}(\mathfrak{sim}(n))$ be the space of algebraic curvature tensors of type $\mathfrak{sim}(n)$, i.e. the space of linear maps from $\Lambda^2 \mathbb{R}^{1,n+1}$ to $\mathfrak{sim}(n)$ satisfying the first Bianchi identity. The curvature tensor $R = R_m$ at the point m belongs to the space $\mathcal{R}(\mathfrak{sim}(n))$. The space $\mathcal{R}(\mathfrak{sim}(n))$ is found in [**10, 12**]. We will review this result now. Fix a null vector $q \in \mathbb{R}^{1,n+1}$ such that $g(p, q) = 1$. Let $E \subset \mathbb{R}^{1,n+1}$ be the orthogonal complement to $\mathbb{R}p \oplus \mathbb{R}q$, then E is an Euclidean space. We get the decomposition

$$\mathbb{R}^{1,n+1} = \mathbb{R}p \oplus E \oplus \mathbb{R}q. \tag{36}$$

We will often write $\mathbb{R}^n$ instead of E. Fixing a basis $X_1,...,X_n$ in $\mathbb{R}^n$, we get that

$$\mathfrak{sim}(n)=\left\{\left.\begin{pmatrix} a & (GX)^t & 0\\ 0 & A & -X\\ 0 & 0 & -a\end{pmatrix}\right| a\in\mathbb{R},\ A\in\mathfrak{so}(n),\ X\in\mathbb{R}^n\right\}, \tag{37}$$

where G is the Gram matrix of the metric $g|_{\mathbb{R}^n}$ with respect to the basis $X_1,...,X_n$. The above matrix can be identified with the triple (a,A,X). We obtain the decomposition

$$\mathfrak{sim}(n)=(\mathbb{R}\oplus\mathfrak{so}(n))\ltimes\mathbb{R}^n.$$

For a subalgebra $\mathfrak{h}\subset\mathfrak{so}(n)$ consider the space

$$\mathcal{P}(\mathfrak{h})=\{P\in(\mathbb{R}^n)^*\otimes\mathfrak{h}|g(P(x)y,z)+g(P(y)z,x)+g(P(z)x,y)=0 \text{ for all } x,y,z\in\mathbb{R}^n\}.$$

Define the map $\widetilde{\mathrm{Ric}}:\mathcal{P}(\mathfrak{h})\to\mathbb{R}^n$, $\widetilde{\mathrm{Ric}}(P)=P^j_{ik}g^{ik}X_j$. It does not depend on the choice of the basis $X_1,...,X_n$. The tensor $R\in\mathcal{R}(\mathfrak{sim}(n))$ is uniquely given by elements $\lambda\in\mathbb{R}, v\in E, R_0\in\mathcal{R}(\mathfrak{so}(n)), P\in\mathcal{P}(\mathfrak{so}(n)), T\in\odot^2E$ in the following way.

$$\begin{aligned} R(p,q)=&(\lambda,0,v), \qquad R(x,y)=(0,R_0(x,y),P(y)x-P(x)y),\\ R(x,q)=&(g(v,x),P(x),T(x)), \qquad R(p,x)=0\end{aligned}$$

for all $x,y\in\mathbb{R}^n$. We write $R=R(\lambda,v,R_0,P,T)$. The Ricci tensor $\mathrm{Ric}(R)$ of R is given by $\mathrm{Ric}(R)(X,Y)=\mathrm{tr}(Z\mapsto R(X,Z)Y)$ and it satisfies

$$\mathrm{Ric}(p,q)=-\lambda,\quad \mathrm{Ric}(x,y)=\mathrm{Ric}(R_0)(x,y), \tag{38}$$

$$\mathrm{Ric}(x,q)=g(x,\widetilde{\mathrm{Ric}}(P)-v),\quad \mathrm{Ric}(q,q)=\mathrm{tr}\,T. \tag{39}$$

Let us take some other null vector q' with $g(p,q')=1$. There exists a unique vector $w\in E$ such that $q'=-\frac{1}{2}g(w,w)p+w+q$. The corresponding E' has the form $E'=\{-g(x,w)p+x|x\in E\}$. We will consider the map $x\in E\mapsto x'=-g(x,w)p+x\in E'$. Using this, we obtain that $R=R(\tilde\lambda,\tilde v,\tilde R_0,\tilde P,\tilde T)$. For example, it holds

$$\tilde\lambda=\lambda,\quad \tilde v=(v-\lambda w)',\quad \tilde P(x')=(P(x)-R_0(x,w))',\quad \tilde R_0(x',y')z'=(R_0(x,y)z)'.$$

This shows that using the change of q we may get rid of v or some times of P. (For example, if $\mathfrak{h}$ is a symmetric Berger algebra, i.e. $\dim\mathcal{R}(\mathfrak{h})=1$, and $R_0\neq0$, then there exists $w\in E$ such that $P(x)-R_0(x,w)=0$ for all x [**12**], i.e. $\tilde P=0$.)

Proof of Theorem 3. Consider the general Walker metric (1). Suppose that it is Einstein with $\Lambda\neq0$. Then $H=\Lambda(x^+)^2+x^+H_1+H_0$, where H_0 and H_1 are independent of x^+ [**15**]. Consider the vector fields

$$p=\partial_+,\quad X_i=\partial_i-A_i\partial_+,\quad q=\partial_--\frac{1}{2}H\partial_+.$$

Let $E\subset TM$ be the distribution generated by the vector fields X_i. At each point m we get

$$T_mM=\mathbb{R}p_m\oplus E_m\oplus\mathbb{R}q_m.$$

Then the curvature tensor R is given by the elements λ,v,R_0,P,T as above but depending on the point. Since the manifold is Einstein, we get $\lambda=-\Lambda$.

Proposition 5. *For any $W\in\Gamma(E)$ such that $\nabla_{\partial_+}W=0$ there exist new Walker coordinates $\tilde x^a$ such that the corresponding vector field q' has the form $q'=-\frac{1}{2}g(W,W)p+W+q$.*

PROOF. Let us write $W=W^iX_i$. Since $\nabla_{\partial_+}W=0$, we get that $\partial_+W^i=0$. We will find the inverse transformation

$$x^+=\tilde x^+,\quad x^i=x^i(\tilde x^1,...,\tilde x^n,\tilde x^-),\quad x^-=\tilde x^-.$$

It holds

$$\tilde\partial_+=\partial_+,\quad \tilde\partial_i=\frac{\partial x^j}{\partial\tilde x^i}\partial_j,\quad \tilde\partial_-=\frac{\partial x^i}{\partial\tilde x^-}\partial_i+\partial_-.$$

For the new Walker metric we have

$$H' = g(\tilde{\partial}_-, \tilde{\partial}_-) = H + 2\frac{\partial x^i}{\partial \tilde{x}^-}A_i + g\left(\frac{\partial x^i}{\partial \tilde{x}^-}\partial_i, \frac{\partial x^j}{\partial \tilde{x}^-}\partial_j\right).$$

Hence,

$$q' = \tilde{\partial}_- - \frac{1}{2}H'\partial_+ = q + U - \frac{1}{2}g(U,U)p,$$

where

$$U = \frac{\partial x^i}{\partial \tilde{x}^-}X_i.$$

The equality $U = W$ is equivalent to the system of equations

$$\frac{\partial x^i(\tilde{x}^1,...,\tilde{x}^n,\tilde{x}^-)}{\partial \tilde{x}^-} = W^i(x^1(\tilde{x}^1,...,\tilde{x}^n,\tilde{x}^-),...,x^n(\tilde{x}^1,...,\tilde{x}^n,\tilde{x}^-),\tilde{x}^-). \tag{40}$$

Consider the system of ordinary differential equations

$$\frac{dy^i(\tilde{x}^-)}{d\tilde{x}^-} = W^i(y^1(\tilde{x}^-),...,y^n(\tilde{x}^-),\tilde{x}^-). \tag{41}$$

Impose the initial conditions $y^i(\tilde{x}^-_0) = \tilde{x}^i$. Then for each set of numbers $\tilde{x}^k$ there exists a unique solution $y^i(x^-)$. Since the solution depends smoothly on the initial conditions, we may write the solution in the form $x^i(\tilde{x}^1,...,\tilde{x}^n,\tilde{x}^-)$. The obtained functions satisfy Equation (40). Since $\det\left(\frac{\partial x^i}{\partial \tilde{x}^j}(\tilde{x}^-_0)\right) \neq 0$, we get that $\det\left(\frac{\partial x^i}{\partial \tilde{x}^j}\right) \neq 0$ for $\tilde{x}^-$ near $\tilde{x}^-_0$. We obtain the required transformation. □

We see that we may choose a Walker coordinate system such that $v = 0$ (if $v \neq 0$, take $W = -\frac{1}{\Lambda}v$, then $\tilde{v} = 0$). It can be shown that

$$v = -\left(\frac{1}{2}\partial_i H_1 - \Lambda A_i\right)h^{ij}X_j.$$

Since $\partial_+\left(\left(\frac{1}{2}\partial_i H_1 - \Lambda A_i\right)h^{ij}\right) = 0$, it holds $\nabla_{\partial_+}W = 0$. Hence we may find a coordinate system, where $A_i = \frac{1}{2\Lambda}\partial_i H_1$. Let us fix this system. In [**15**] it is noted that under the transformation

$$\tilde{x}^+ = x^+ - f(x^1,...,x^n,x^-), \quad \tilde{x}^i = x^i, \quad \tilde{x}^- = x^-$$

the metric (1) changes in the following way

$$A_i \mapsto A_i + \partial_i f, \quad H_1 \mapsto H_1 + 2\Lambda f, \quad H_0 \mapsto H_0 + H_1 f + \Lambda f^2 + 2\dot{f}. \tag{42}$$

Thus if we take $f = -\frac{1}{2\Lambda}H_1$, then with respect to the new coordinates we have $A_i = H_1 = 0$. Now Theorem 3 follows from (4–7). □

Proof of Proposition 2. The decomposition of the $\mathfrak{so}(n)$-projection of the holonomy as in (24), $\mathfrak{h} = \{0\} \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$ defines parallel distributions $E^0, \ldots, E^r$, all containing the parallel distribution of null lines. These distributions, in turn, define coordinates

$$x^+, x^1_0, \ldots, x^{n_0}_0, \ldots, x^1_r, ..., x^{n_r}_r, x^-$$

such that E^α is spanned by $\partial_+, \frac{\partial}{\partial x^1_\alpha}, \ldots, \frac{\partial}{\partial x^{n_\alpha}_\alpha}$ and such that they are adapted in the sense of Section 2. Note that the most general coordinate transformation preserving these properties is given by

$$\begin{array}{rcl} \tilde{x}^+ & = & x^+ + \varphi(x^1_0, \ldots, x^{n_r}_r, x^-), \\ \tilde{x}^i_0 & = & \sum_{j=1}^{n_0} a^i_j x^j_0 + b^i, \quad \text{for } i = 1, \ldots, n_0, \\ \tilde{x}^i_\alpha & = & \psi^i_\alpha(x^1_\alpha, \ldots, x^{n_\alpha}_\alpha, x^-), \quad \text{for } i = 1, \ldots, n_\alpha \text{ and } \alpha = 1, \ldots, r, \\ \tilde{x}^- & = & x^- + c, \end{array} \tag{43}$$

here $\frac{\partial^2}{\partial x^j_\beta \partial x^i_\alpha}\varphi = 0$ if $\beta \neq \alpha$, $(a^i_j)^{n_0}_{i,j=1}$ is an orthogonal matrix and $b^i \in \mathbb{R}$. Choosing $\varphi \equiv 0$, it is clear that Equation (31) can be solved separately for each $\alpha = 1, \ldots, r$. This shows that the coordinates found in Theorem 1 can be chosen to be adapted.

Now we turn to the second statement of Proposition 2. Let us assume that $\Lambda\neq 0$. Starting with adapted coordinates, Equation (5) shows that

$$\frac{\partial^2}{\partial x^j_\beta \partial x^i_\alpha} H_1 = 0, \quad \text{if } \beta\neq\alpha. \tag{44}$$

Consider the proof of Theorem 3 applied to a metric in adapted coordinates in order to prove the second statement. Equation (7) shows that $n_0=0$. Recall that we consider the system of equations (40) for $W^i=\frac{1}{\Lambda}\left(\frac{1}{2}\partial_j H_1-\Lambda A_j\right)h^{ij}$. Since we have the property (44), we get that if the index i corresponds to the space $\mathbb{R}^{n_\alpha}$, then $\frac{\partial}{\partial x^k_\beta}W^i=0$ if $\beta\neq\alpha$. It is obvious that we get r independent systems of equations, each of these systems is a system with respect to the unknown functions $x^1_\alpha(\tilde{x}^1_\alpha,...,\tilde{x}^{n_\alpha}_\alpha),...,x^{n_\alpha}_\alpha(\tilde{x}^1_\alpha,...,\tilde{x}^{n_\alpha}_\alpha)$. It is clear that the solution for such a system obtained above satisfies the requirements of the proposition. $\square$

Proof of Proposition 3. As above, let $R=R(\lambda,v,R_0,P,T)$. Consider the coordinate system as in Theorem 3. Then, $v=0$ and $\widetilde{\mathrm{Ric}}(P)=0$. The decomposition (24) implies $P=P_1+\cdots+P_r$, where $P_\beta\in\mathcal{P}(\mathfrak{h}_\beta)$. Consequently, each $\widetilde{\mathrm{Ric}}(P_\beta)$ is zero. Since $\mathfrak{h}_\alpha\subset\mathfrak{so}(n_\alpha)$ is a symmetric Berger algebra, the equality $\widetilde{\mathrm{Ric}}(P_\alpha)=0$ implies $P_\alpha=0$ [**12**], and this is exactly Equation (25). $\square$

4. Examples

Suppose that metric (1) is Einstein with the cosmological constant $\Lambda\neq 0$. Then (3) holds. According to Theorem 3, there exist new Walker coordinates $(\tilde{x}^+,\tilde{x}^1,...,\tilde{x}^n,\tilde{x}^-)$ such that $\tilde{A}=0$ and $\tilde{H}_1=0$. The proof of Theorem 3 implies that such coordinates can be found in the following way. Consider the system of ordinary differential equations

$$\frac{dy^i(\tilde{x}^-)}{d\tilde{x}^-}=W^i(y^1(\tilde{x}^-),...,y^n(\tilde{x}^-),\tilde{x}^-), \tag{45}$$

where $W^i=(\frac{1}{2\Lambda}\partial_j H_1-A_j)h^{ij}$ and impose the initial conditions $y^i(\tilde{x}^-_0)=\tilde{x}^i$. This will give the inverse transformation

$$x^+=\tilde{x}^+,\quad x^i=x^i(\tilde{x}^1,...,\tilde{x}^n,\tilde{x}^-),\quad x^-=\tilde{x}^-$$

and allow to find the metric with respect to the new coordinates. Note that $\tilde{H}_1=H_1$. If $H_1=0$, then with respect to the obtained coordinates $\tilde{A}_i=\tilde{H}_1=0$ holds. If $H_1\neq 0$, then it is necessary to consider the additional transformation

$$\tilde{x}^+\mapsto\tilde{x}^++\frac{1}{2\Lambda}H_1,\quad \tilde{x}^i\mapsto\tilde{x}^i,\quad \tilde{x}^-\mapsto\tilde{x}^-.$$

After this $\tilde{A}_i=\tilde{H}_1=0$.

The required coordinates can be found also in the following way. First consider the transformation

$$x^+\mapsto x^++\frac{1}{2\Lambda}H_1,\quad x^i\mapsto x^i,\quad x^-\mapsto x^-.$$

After this $H_1=0$ and A_i changes to $A_i-\frac{1}{2\Lambda}\partial_i H_1$. After this consider the system of ordinary differential equations (45) with $W^i=-A_jh^{ij}$ and impose the initial conditions $y^i(\tilde{x}^-_0)=\tilde{x}^i$. With respect to the obtained coordinates $\tilde{A}_i=\tilde{H}_1=0$ holds.

For $n=2$ and $\Lambda\neq 0$ all solutions to Equation (2) for metric (1) are obtained in [**23**]. It is proved that any such metric is given in the following way (we use slight modifications). There exist coordinates x^+,u,v,x^- such that

$$g=\frac{2}{P^2}\mathrm{d}z\mathrm{d}\bar{z}+\left(2\mathrm{d}x^++2W\mathrm{d}z+2\bar{W}\mathrm{d}\bar{z}+\left(\Lambda\cdot(x^+)^2+H_0\right)\mathrm{d}x^-\right)\mathrm{d}x^-,$$

where

$$z=u+iv,\quad 2P^2=|\Lambda|2P_0^2=|\Lambda|\left(1+\frac{\Lambda}{|\Lambda|}z\bar z\right)^2,\quad W=i\partial_z L,$$
$$L=2\mathrm{Re}\left(f\partial_z(\ln P_0)-\frac{1}{2}\partial_z f\right),$$

$f=f(z,x^-)$ is an arbitrary function holomorphic in z and smooth in x^-, the function $H_0=H_0(z,\bar z,x^-)$ can be expressed in a similar way in terms of f and another arbitrary function holomorphic in z and smooth in x^-.

Using this result, we consider several examples.

Example 1. Let $\Lambda<0$ and $f=c(x^-)$, we obtain the following metric

$$g=2\mathrm{d}x^+\mathrm{d}x^-+\frac{4}{-\Lambda\cdot(1-u^2-v^2)^2}\big((\mathrm{d}u)^2+(\mathrm{d}v)^2\big)$$
$$+\frac{c(x^-)}{(1-u^2-v^2)^2}\big(-4uv\mathrm{d}u+2(u^2-v^2+1)\mathrm{d}v\big)\mathrm{d}x^-+(\Lambda\cdot(x^+)^2+H_0)(\mathrm{d}x^-)^2,$$

which becomes Einstein after a proper choice of the function H_0. Equations (45) take the form

$$\frac{\partial u}{\partial\tilde x^-}=-\frac{\Lambda}{2}uvc(x^-),\qquad \frac{\partial v}{\partial\tilde x^-}=\frac{\Lambda}{4}(u^2-v^2+1)c(x^-).$$

Using Maple 12, we find that the general solution of this system has the form

$$u=\frac{64c_1\Lambda^2}{\left(c_1^2\left(4e^{-\frac{1}{2}\Lambda b(\tilde x^-)}+\Lambda c_2\right)^2+64\Lambda^4\right)e^{\frac{1}{2}\Lambda\tilde b(x^-)}},$$
$$v=\frac{-16c_1^2e^{-\Lambda b(\tilde x^-)}+c_1^2c_2^2\Lambda^2+64\Lambda^4}{c_1^2\left(4e^{-\frac{1}{2}\Lambda b(\tilde x^-)}+\Lambda c_2\right)^2+64\Lambda^4},$$

where c_1 and c_2 are arbitrary functions of $\tilde u$ and $\tilde v$, $b(\tilde x^-)$ is the function such that $\frac{\mathrm{d}b(\tilde x^-)}{\mathrm{d}\tilde x^-}=c(\tilde x^-)$ and $b(0)=0$. Substituting the initial conditions $u(0)=\tilde u$, $v(0)=\tilde v$, we obtain

$$c_1=\frac{\tilde u^2+\tilde v^2-2\tilde v^2+1}{\tilde u}\Lambda^2,\quad c_2=-4\frac{\tilde u^2+\tilde v^2-1}{\Lambda\cdot(\tilde u^2+\tilde v^2-2\tilde v^2+1)}.$$

With respect to the obtained coordinates, we get

$$g=2\mathrm{d}x^+\mathrm{d}x^-+\frac{4}{-\Lambda\cdot(1-u^2-v^2)^2}\big((\mathrm{d}u)^2+(\mathrm{d}v)^2\big)+(\Lambda\cdot(x^+)^2+\tilde H_0)(\mathrm{d}x^-)^2.\tag{46}$$

The metric g is Einstein if and only if $(\partial_u^2+\partial_v^2)\tilde H_0=0$. Taking sufficiently general solutions of this equation (e.g. $\tilde H_0=uv$), we obtain that this metric is indecomposable and its holonomy algebra is isomorphic to $(\mathbb{R}\oplus\mathfrak{so}(2))\ltimes\mathbb{R}^2$.

Note that taking $f=z^2$, one obtains the same example.

Example 2. Let $\Lambda<0$ and $f=zc(x^-)$, we obtain the following metric

$$g=2\mathrm{d}x^+\mathrm{d}x^-+\frac{4}{-\Lambda\cdot(1-u^2-v^2)^2}\big((\mathrm{d}u)^2+(\mathrm{d}v)^2\big)$$
$$+\frac{2c(x^-)}{(1-u^2-v^2)^2}\big(v\mathrm{d}u-u\mathrm{d}v\big)\mathrm{d}x^-+(\Lambda\cdot(x^+)^2+H_0)(\mathrm{d}x^-)^2.$$

Equations (45) take the form

$$\frac{\partial u}{\partial\tilde x^-}=\frac{\Lambda}{4}vc(x^-),\qquad \frac{\partial v}{\partial\tilde x^-}=-\frac{\Lambda}{4}uc(x^-).$$

The general solution of this system has the form

$$u=c_1\cos\left(\frac{\Lambda}{4}b(\tilde x^-)\right)+c_2\sin\left(\frac{\Lambda}{4}b(\tilde x^-)\right),\quad v=-c_1\sin\left(\frac{\Lambda}{4}b(\tilde x^-)\right)+c_2\cos\left(\frac{\Lambda}{4}b(\tilde x^-)\right),$$

where c_1 and c_2 are arbitrary functions of $\tilde{u}$ and $\tilde{v}$, and $b(\tilde{x}^-)$ is the function such that $\frac{\mathrm{d}b(\tilde{x}^-)}{\mathrm{d}\tilde{x}^-} = c(\tilde{x}^-)$ and $b(0) = 0$. Substituting the initial conditions $u(0) = \tilde{u}$, $v(0) = \tilde{v}$, we obtain $c_1 = \tilde{u}$, $c_2 = \tilde{v}$. With respect to the obtained coordinates, we again get

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + \frac{4}{-\Lambda\cdot(1-u^2-v^2)^2}\big((\mathrm{d}u)^2+(\mathrm{d}v)^2\big) + (\Lambda\cdot(x^+)^2+\tilde{H}_0)(\mathrm{d}x^-)^2. \tag{47}$$

Example 3. Let $\Lambda > 0$ and $f = zc(x^-)$, we obtain the following metric

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + \frac{4}{\Lambda\cdot(1+u^2+v^2)^2}\big((\mathrm{d}u)^2+(\mathrm{d}v)^2\big) \\ + \frac{2c(x^-)}{(1+u^2+v^2)^2}\big(v\mathrm{d}u - u\mathrm{d}v\big)\mathrm{d}x^- + (\Lambda\cdot(x^+)^2+H_0)(\mathrm{d}x^-)^2.$$

Equations (45) take the form

$$\frac{\partial u}{\partial\tilde{x}^-} = -\frac{\Lambda}{4}vc(x^-), \quad \frac{\partial v}{\partial\tilde{x}^-} = \frac{\Lambda}{4}uc(x^-).$$

The general solution of this system has the form

$$u = c_1\cos\left(\frac{\Lambda}{4}b(\tilde{x}^-)\right) + c_2\sin\left(\frac{\Lambda}{4}b(\tilde{x}^-)\right), \quad v = c_1\sin\left(\frac{\Lambda}{4}b(\tilde{x}^-)\right) - c_2\cos\left(\frac{\Lambda}{4}b(\tilde{x}^-)\right),$$

where c_1 and c_2 are arbitrary functions of $\tilde{u}$ and $\tilde{v}$, and $b(\tilde{x}^-)$ is the function such that $\frac{\mathrm{d}b(\tilde{x}^-)}{\mathrm{d}\tilde{x}^-} = c(\tilde{x}^-)$ and $b(0) = 0$. Substituting the initial conditions $u(0) = \tilde{u}$, $v(0) = \tilde{v}$, we obtain $c_1 = \tilde{u}$, $c_2 = -\tilde{v}$. With respect to the obtained coordinates, we get

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + \frac{4}{\Lambda\cdot(1+u^2+v^2)^2}\big((\mathrm{d}u)^2+(\mathrm{d}v)^2\big) + (\Lambda\cdot(x^+)^2+\tilde{H}_0)(\mathrm{d}x^-)^2. \tag{48}$$

The metric g is Einstein if and only if $(\partial_u^2+\partial_v^2)\tilde{H}_0 = 0$. Taking sufficiently general solution of this equation (e.g. $\tilde{H}_0 = uv$), we obtain that this metric is indecomposable and its holonomy algebra is isomorphic to $(\mathbb{R}\oplus\mathfrak{so}(2))\ltimes\mathbb{R}^2$.

For most of the other functions f Equations (40) and their solutions become much more difficult. Further examples are considered in [**14**]. In particular, in [**14**] there are obtained examples such that the Riemannian part h depends non-trivially on the parameter x^-.

Consider the general Walker metric (1). Theorem 1 shows that there exist coordinates $(\tilde{x}^+, \tilde{x}^1, ..., \tilde{x}^n, \tilde{x}^-)$ such that $\tilde{A} = 0$. These coordinates can be found as in the proof of Theorem 1 or in the following alternative way.

Consider the transformation given by the inverse one $x^+ = \tilde{x}^+$, $x^i = x^i(\tilde{x}^1, ..., \tilde{x}^n, \tilde{x}^-)$, $x^- = \tilde{x}^-$. It holds

$$\tilde{\partial}_+ = \partial_+, \quad \tilde{\partial}_i = \frac{\partial x^j}{\partial\tilde{x}^i}\partial_j, \quad \tilde{\partial}_- = \frac{\partial x^i}{\partial\tilde{x}^-}\partial_i + \partial_-.$$

For the new Walker metric we get

$$\tilde{A}_i = \frac{\partial x^j}{\partial\tilde{x}^i}\left(A_j + h_{jk}\frac{\partial x^k}{\partial\tilde{x}^-}\right).$$

Hence, if the equalities

$$\frac{\partial x^i}{\partial\tilde{x}^-} = -A_jh^{ji} \tag{49}$$

hold, then $\tilde{A}_i = 0$. Impose the conditions $x^i(\tilde{x}^1, ..., \tilde{x}^n, \tilde{x}^-_0) = \tilde{x}^i$. Then for each set of numbers $\tilde{x}^k$ there exists a unique solution $x^i(x^-)$ of the above system of equations. Since the solution depends smoothly on the initial conditions, we may write the solution in the form $x^i(\tilde{x}^1, ..., \tilde{x}^n, \tilde{x}^-)$. The obtained functions satisfy Equation (49). Since $\det\left(\frac{\partial x^i}{\partial\tilde{x}^j}(\tilde{x}^-_0)\right) \neq 0$, we get that $\det\left(\frac{\partial x^i}{\partial\tilde{x}^j}\right) \neq 0$ for $\tilde{x}^-$ near $\tilde{x}^-_0$. We obtain the required transformation.

Ricci-flat Walker metrics in dimension 4 are found in [**18, 19**]. They are of the form

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + (\mathrm{d}u)^2 + (\mathrm{d}v)^2 + 2A_1\mathrm{d}x\mathrm{d}x^- + (-(\partial_u A_1)x^+ + H_0)(\mathrm{d}x^-)^2, \tag{50}$$

where A_1 and H_0 satisfy $\partial_+ A_1 = \partial_+ H_0 = 0$,

$$\partial_u^2 A_1 + \partial_v^2 A_1 = 0, \tag{51}$$

$$\partial_u^2 H_0 + \partial_v^2 H_0 = 2\partial_-\partial_u A_1 - 2A_1\partial_u^2 A_1 - (\partial_u A_1)^2 + (\partial_v A_1)^2. \tag{52}$$

Note that in order to get rid of the function A_1 it is enough to consider the transformation with the inverse one

$$x^+ = \tilde{x}^+,\ u = f(\tilde{u}, \tilde{v}, \tilde{x}^-),\ v = \tilde{v},\ x^- = \tilde{x}^-$$

such that the function f satisfies the equation

$$\partial_- f(\tilde{u}, \tilde{v}, \tilde{x}^-) = -A_1(f(\tilde{u}, \tilde{v}, \tilde{x}^-), \tilde{v}, \tilde{x}^-). \tag{53}$$

Imposing the condition $f(\tilde{u}, \tilde{v}, 0) = \tilde{u}$, we may consider the coordinates $\tilde{u}$ and $\tilde{v}$ as the parameters, then the obtained equation is an ordinary differential equation.

Example 4. It is clear that $A_1 = uv$ and $H_0 = \frac{1}{12}(u^4 - v^4)$ are solutions of (51) and (52). We get the following Ricci-flat metric:

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + (\mathrm{d}u)^2 + (\mathrm{d}v)^2 + 2uv\mathrm{d}u\mathrm{d}x^- + \left(-vx^+ + \frac{1}{12}(u^4 - v^4)\right)(\mathrm{d}x^-)^2. \tag{54}$$

Equation (53) takes the form

$$\partial_- f(\tilde{u}, \tilde{v}, \tilde{x}^-) = -f(\tilde{u}, \tilde{v}, \tilde{x}^-)\tilde{v}$$

and it defines the transformation

$$\tilde{x}^+ = x^+,\ \tilde{u} = ue^{vx^-},\ \tilde{v} = v,\ \tilde{x}^- = x^-.$$

With respect to the obtained coordinates, we get

$$\begin{aligned} g = {} & 2\mathrm{d}x^+\mathrm{d}x^- + e^{-2vx^-}(\mathrm{d}u)^2 - 2ux^-e^{-2vx^-}\mathrm{d}u\mathrm{d}v + \left(1 + u^2(x^-)^2e^{-2vx^-}\right)(\mathrm{d}v)^2 \\ & + \left(-vx^+ - u^2v^2e^{-2vx^-} - \frac{1}{12}v^4 + \frac{1}{12}u^4e^{-4x^-v}\right)(\mathrm{d}x^-)^2. \end{aligned} \tag{55}$$

The holonomy algebra of this metric equals to $(\mathbb{R} \oplus \mathfrak{so}(2)) \ltimes \mathbb{R}^2$.

Example 5. The functions $A_1 = e^u \cos v$ and $H_0 = -\frac{1}{4}(1 + 2v\sin 2v)e^{2u}$ are solutions of (51) and (52). We get the following Ricci-flat metric:

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + (\mathrm{d}u)^2 + (\mathrm{d}v)^2 + 2e^u\cos v\mathrm{d}u\mathrm{d}x^- + \left(-x^+e^u\cos v - \frac{1}{4}(1 + 2v\sin 2v)e^{2u}\right)(\mathrm{d}x^-)^2. \tag{56}$$

Equation (53) takes the form

$$\partial_- f(\tilde{u}, \tilde{v}, \tilde{x}^-) = -e^{f(\tilde{u}, \tilde{v}, \tilde{x}^-)}\cos\tilde{v}$$

and it defines the transformation

$$\tilde{x}^+ = x^+,\ \tilde{u} = -\ln\left(e^{-u} - x^-\cos v\right),\ \tilde{v} = v,\ \tilde{x}^- = x^-.$$

With respect to the obtained coordinates, we get

$$\begin{aligned} g = {} & 2\mathrm{d}x^+\mathrm{d}x^- + \frac{1}{\left(x^-e^u\cos v + 1\right)^2}\left((\mathrm{d}u)^2 + 2x^-e^u\sin v\mathrm{d}u\mathrm{d}v + \left(1 + x^-e^u\right)(\mathrm{d}v)^2\right) \\ & - \frac{1}{4\left(x^-e^u\cos v + 1\right)^2}\left(4x^+\left(x^-\cos^2 v + e^{-u}\cos v\right) + 1 + 4\cos^2 v + 2v\sin 2v\right)(\mathrm{d}x^-)^2. \end{aligned} \tag{57}$$

The holonomy algebra of this metric equals to $(\mathbb{R} \oplus \mathfrak{so}(2)) \ltimes \mathbb{R}^2$.

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